THE

# THEORY OF ELASTICITY

APPLIED TO A

# SYSTEM OF CONTINUOUS GIRDERS AND COLUMNS.

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BY

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# INTRODUCTION.

THIS pamphlet proposes to illustrate, by an example, how the theory of elasticity may be used for determining stresses in complex cases, which at first sight would seem unfit for rational treatment.

It would be impossible to produce in so limited space a complete treaty of the elasticity in structures; however, by every advanced student, architect or engineer, the example, here given, will not only be easily understood, but it may show them the way, how in any other case, the designer may enter into the causes of the different strains and, following them out, may make reasonable assumptions, which will easily lead them to true results.

Such assumptions might be objected, but is not the theory of flexure a mere reasonable assumption? is not the assumption of a theoretical hinge at the connecting points of truss-members an ideal? In a calculation the designer may make any reasonable assumptions, provided that he is aware of it and takes in account all their consequences.

The example here chosen is one that happens more frequently than it would seem. In fact, every system of girders and columns, connected solidly to one piece by means of rivets, presents such a case. A row of rivets connecting a beam or girder to a column, even if only intended to transmit shearing-stresses, will necessarily transmit a moment and generally as much as the total moment, which the load on the beam or girder could possibly exert on the column by means of big gusset-plates; because in practice the rivets are only used to one-fifth to one-eighth of their total strength, but in bending the columns they act every time with their full and ultimate capacity.

No doubt a great many accidents are due rather to the bad design of connections and the superficial computation of the strains, based on unreasonable assumptions, than to bad quality of material.

CHAS. STEINER.

# THE INFLUENCE OF LIVE LOAD ON A SYSTEM OF CONTINUOUS GIRDERS AND COLUMNS.

While in bridge construction continuous girders have been succeeded by the cantilever system, even in those countries where they were formerly used to a very large extent, still in buildings a system of continuous columns and girders, riveted together in one piece, is often used.

Provided that the foundation is an absolutely good, resistant one, and that proper care is taken in the calculation of stresses, as they really occur, especially with respect to the live load, such structures are cheaper and as good as others.

The following deductions are made for an example, which frequently occurs, and for special cases a parallel may easily be drawn by the attentive reader. It will be useful to recall three considerations with respect to simpler cases:

I. A concentrated load at the end of a beam, sticking at the

Fig. 1.)

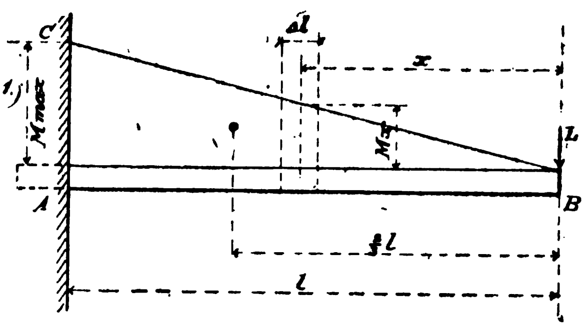
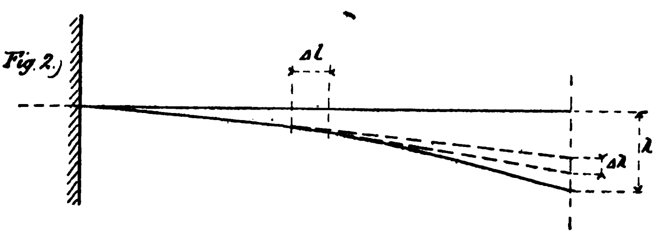



Fig. 2.)




other end in a wall. The load $L$ (Fig. 1) at the end of a beam of the length $l$, will produce at the wall a maximum moment shown

in a certain scale as $M_{max}$, the intermediate moments diminishing in straight line to zero at the end; the elastic line (Fig. 2), drawn in exaggerated scale, is formed by the deflection of the elements. The deflection of an element $\Delta l$ at the distance $x$ is determined by the formulae:

$$tg\,\Delta\delta = \frac{M_x \Delta l}{E \times J},$$

where $\Delta\delta$ the angle of deflection, $\Delta l$ the length of the element, $E$ the coefficient of elasticity, and $J$ the moment of inertia, which, in the following, is considered constant.

The vertical movement of the end $B$, due to the deflection of this element, is therefore:

$$\Delta\lambda = \frac{M_x \Delta l}{E \times J} \times x$$

The total vertical movement is the sum, or

$$\lambda = \Sigma\Delta\lambda = \frac{1}{EJ}\Sigma M_x \Delta l\, x$$

The value $\Sigma\, x\, M_x\, \Delta l$ is nothing else than the static moment of triangle $A, B, C$, with respect to a vertical through $B$, or equal to the area of $A, B, C$, multiplied by $\frac{2}{3}l$; hence:

$$\lambda = \frac{1}{EJ} \times \frac{M_{max}}{2} \times l \times \tfrac{2}{3}l = \frac{L \times l^3}{3EJ};$$

$M_{max}$ being equal to $L \times l$.

We find likewise for the total angle of deflection:

$$tg\,\delta = \Sigma\frac{M_x \Delta l}{EJ} = \frac{L \times l}{2EJ};$$

$\Sigma M_x \Delta l$ being equal to $\dfrac{L \times l}{2}$, and from the two values for $\lambda$ and $tg\,\delta$ the relation:

$$L \times l = 2EJ\,tg\,\delta = \frac{3EJ\lambda}{l^2};\ \text{hence:}$$

$$\lambda = \frac{2\,l^2\,tg\,\delta}{3}.$$

II. A uniformly distributed load $p$, per lin. foot on a straight beam, sticking horizontally in the walls at both ends. (Figs. 3

and 4). If $M_{max}$ is the maximum moment for a free beam, we

Fig. 3.)

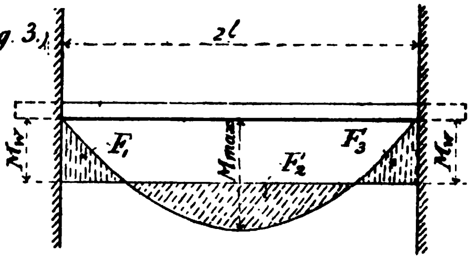
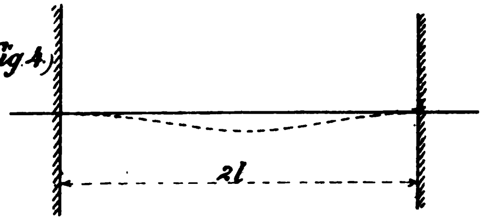



Fig. 4.)




find the real moment $M_r$ at center and the moments $M_w$ at the wall by the following consideration :

The deflection of one element of length is determined by :

$$tg\,\Delta\,\delta = \frac{M_x\,\Delta\,l}{E\,J},$$

hence the total deflection, positive and negative, over the whole length $2\,l$, will be :

$$tg\,\delta = \Sigma\,tg\,\Delta\,\delta = \Sigma\frac{M_x\,\Delta\,l}{E\,J},$$

which, as the end-tangents are same, must equal zero.

$E\,J$ is a constant, hence : $\Sigma\,M_x\,\Delta\,l = 0$. But $M_x\,\Delta\,l$ is nothing else than the algebraic sum of the three areas $F_1$, $F_2$, $F_3$, cut from the above parabola ; $F_1$ and $F_3$ are negative and $F_2$ positive, hence :

$$F_1 + F_3 = F_2, \text{ or}$$

$$\tfrac{2}{3}\,M_{max} \times 2l - M_w \times 2l = 0$$

$$M_w = \tfrac{2}{3}\,M_{max}.$$

III. Assuming that a beam is sticking in the walls at both ends, or otherwise solidly connected to columns or solid bodies $A$ and $B$ (Fig. 5), which under a uniform load of the beam, give way to a certain extent, according to their capacity and strength, so as to form the angles $\alpha$ and $\beta$, then the end-tangents of the elastic line form the angle $\alpha + \beta$ ; therefore the total deflection of the beam

$A, B$ (positive and negative summed up), due to the moments, acting on the beam is:

Fig. 5.)

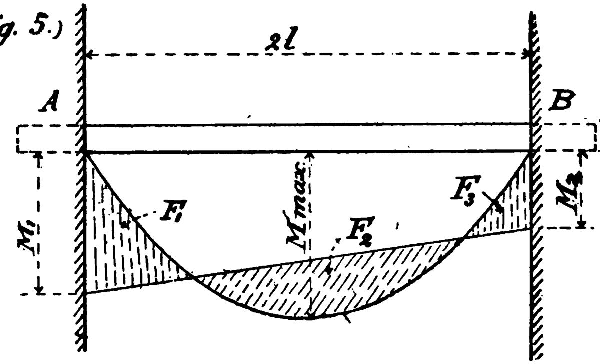



Fig. 6.)

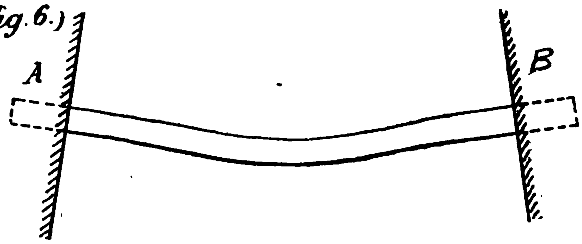
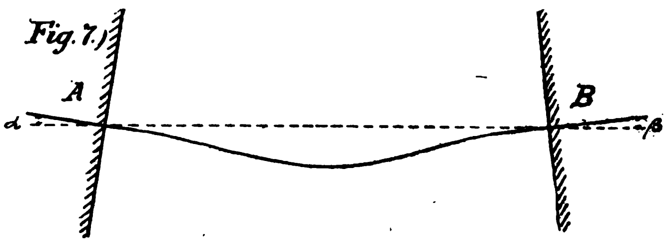



Fig. 7.)




$$\Sigma\, tg\, \Delta\, \delta = \Sigma \frac{M_x \Delta\, l}{E J} = tg\,(\alpha + \beta),$$

or, as the angles are very small:

$$\Sigma \frac{M_x \Delta\, l}{E J} = tg\, \alpha + tg\, \beta\,; \; EJ \text{ being constant:}$$

$$\Sigma\, M_x \Delta\, l = (tg\, \alpha + tg\, \beta)\, E J, \text{ hence,}$$

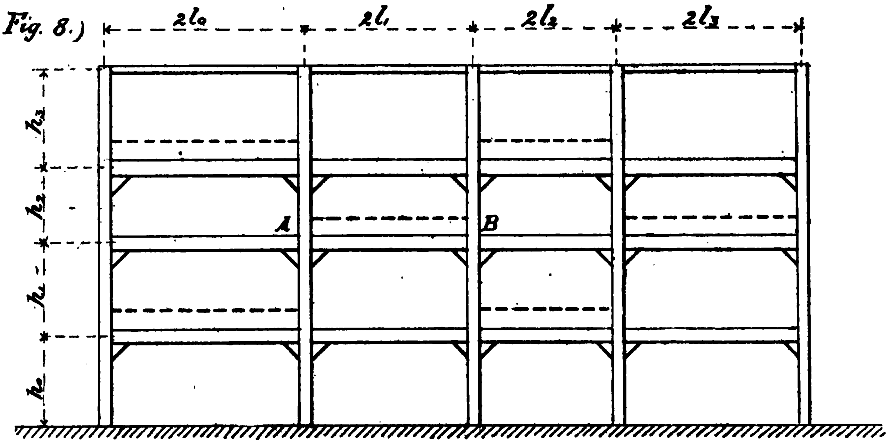
$$\tfrac{2}{3}\, M_{max} \times 2l - \frac{M_1 + M_2}{2} \times 2l = (tg\, \alpha + tg\, \beta)\, E J, \text{ or}$$

$$M_1 + M_2 = 2 \left[ \tfrac{2}{3}\, M_{max} - (tg\, \alpha + tg\, \beta) \frac{E J}{2l} \right]$$

*Remark.*—In the formulaes under I, II and III the influences of the shearing forces are neglected, and only the deformations due to the moments are considered. This will be an allowed abbreviation, the former being inconsiderable in plate-girders and columns with web-plates, which are here exclusively employed.

*Application.*—Let Fig. 8 represent a building consisting of riveted columns and girders, solidly riveted to one piece, and the connecting points strong enough to make the whole system con-

*Fig. 8.)*




tinuous, both in a vertical and horizontal direction. In order to find the most dangerous stresses from live load in a portion $A$, $B$, and adjacent columns, let every second panel of each story be loaded and the others unloaded, as indicated by broken lines; the load being supposed uniformly distributed.

The deformations of girders and columns will take place, as shown by Fig. 9, the angles at the connections remaining the same as before, say 90°.

The columns at $A$ and $B$ will allow by their elasticity a certain deflection of the end tangents of the beam, say $\alpha$ and $\beta$, as shown in above No. III. The angle of deflection $\alpha$ in point $A$ is limited by the resistance of the two columns $A$, $C$ and $A$, $E$, and the girder $A$, $G$, which is here supposed unloaded. In order to get a judgment about these resistances some reasonable assumptions are made, which the reader is invited to check himself:

1. The material in columns $A$, $C$ and $A$, $E$, and girders $A$, $B$ and $J$, $C$, is supposed to be strained to the same amount per square inch by the maximum live loads $p_1$ and $p_2$, etc.

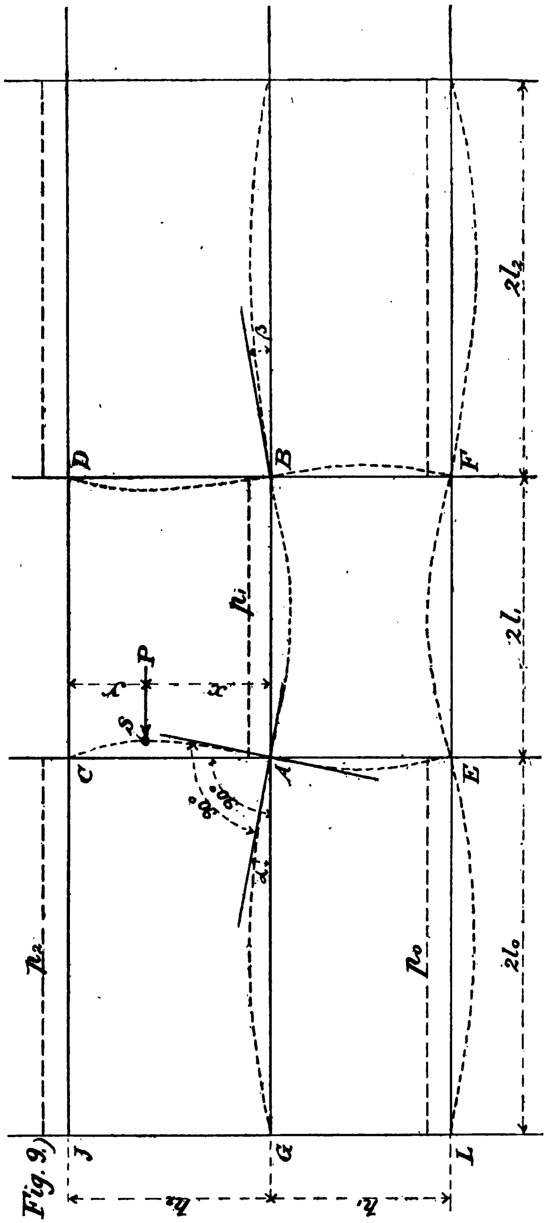



Fig. 9)

2. If $p_1$ equals $p_2$ the influence on columns $A$, $C$, due to girder $A$, $B$ and girder $J$, $C$, will be in direct proportion to the max. moment of these girders, considered free, hence in direct proportion to the squares of $l_1$ and $l_0$.

3. If the spans $2l_0 = 2l_1$, the influence of these girders on column $A$, $C$ will be in direct proportion to $p_1$ and $p_2$.

4. If $p_2$ unequal $p_1$ and $2l_0$ unequal $2l_1$, the influences of $A$, $B$ and $J$, $C$ on column $A$, $C$ will therefore be in the proportion of $p_1 l_1^2 : p_2 l_0^2$.

5. If $G$ is a point of an end-column the influence of girders $J$, $C$ and $L$, $E$ on point $G$, and consequently on girder $G$, $A$, will be to the influence of $A$, $B$ on $G$, $A$ in the proportion of : $p_2 l_0^2 + p_0 l_0^2 : p_1 l_1^2$, which, for the portion in view, will be on the safe side (giving largest values for the moment at center of girder and for bending moments of columns; for the max. moments in the girder at columns, the total load on all spans has to be considered).

On account of assumption 4, there will be, for the general case, a point $S$ on the elastic line of $A$, $C$, (Fig. 9), wherein an unknown force $P$ will produce the bending-moments at $A$ and $C$, which correspond to the assumed loads. As the moment in point $S$ is zero, the tangent of the elastic line is vertical in this point. The bending-moments $Px$ and $Py$ are the above influences of No. 4, and the location of $P$ must, therefore, be determined by the reversed proportion :

$$\frac{x}{y} = \frac{\text{influence of girder } J,\ C}{\text{influence of girder } A,\ B} = \frac{p_2 l_0^2}{p_1 l_1^2};$$

wherefrom $x$ as well as $y$ may be determined by means of the other equation, $x + y = h_2$.

Similar relations we find for the unknown forces and the constant levers for $A$, $E$ and $A$, $G$, etc., (as shown in Fig. 10).

In order to find the real bending-moments $M_1$ and $M_2$, at the end of the girder $A$, $B$, as well as the different bending-moments on the colnmns and adjacent girders, such as $P_1 \lambda_1$, $V_3 k_3$, etc., (Fig. 10), we can, according to preceding No. I and III, dispose of the following equations, where the different $J_0$, $J_3$, $J\alpha'$, $J\beta'$, $J\alpha''$, $J\beta''$, etc., are the moments of inertia, where indicated.

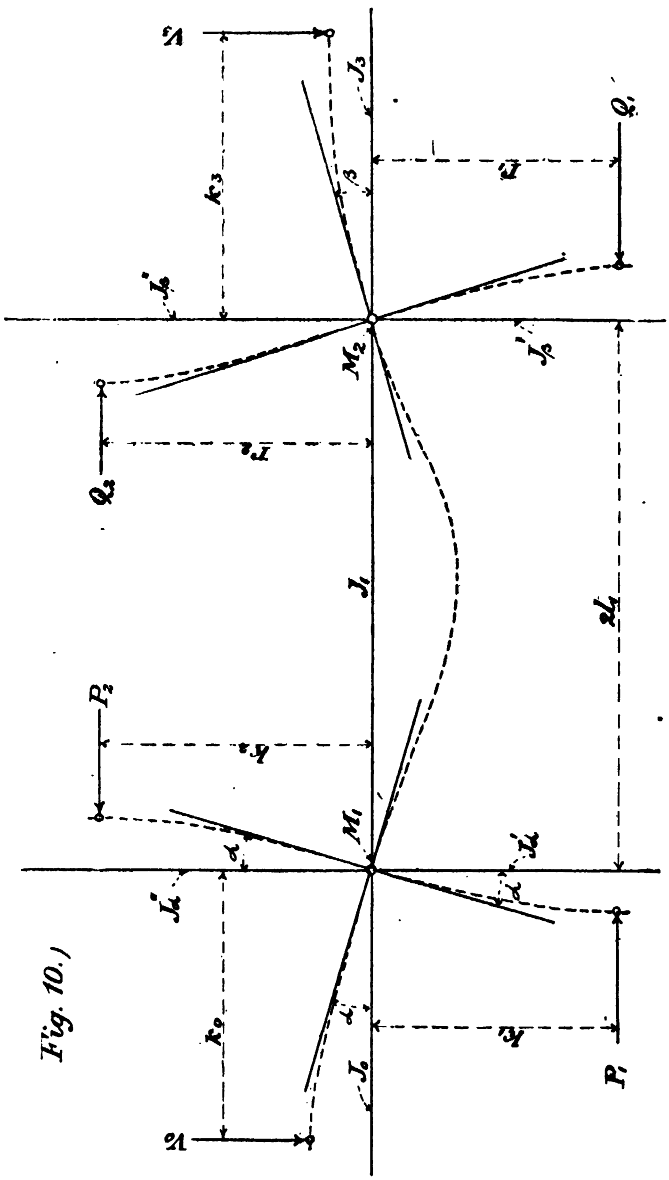



Fig. 10.)

From No. I: $P_1 = \frac{3 E J_\alpha' \lambda_1}{k_1^3}$; also:

$\lambda_1 = \frac{2}{3} k_1^2 \, tg\, \alpha$; from these two equations:

$P_1 k_1 = \frac{2}{3} k_1^2 \, tg\, \alpha \times \frac{3 E J_\alpha'}{k_1^2}$; wherefrom:

1. $P_1 k_1 = 2 E J_\alpha' \, tg\, \alpha$; likewise:
2. $P_2 k_2 = 2 E J_\alpha'' \, tg\, \alpha.$
3. $V_0 k_0 = 2 E J_0 \, tg\, \alpha.$
4. $M_1 \;\; = P_1 k_1 + P_2 k_2 + V_0 k_0.$
5. $Q_1 r_1 = 2 E J\beta' \, tg\, \beta.$
6. $Q_2 r_2 = 2 E J\beta'' \, tg\, \beta.$
7. $V_3 k_3 = 2 E J_3 \, tg\, \beta.$
8. $M_2 \;\; = Q_1 r_1 + Q_2 r_2 + V_3 k_3$; furthermore, from No. III:
9. $M_1 + M_2 = 2 \left[ \frac{2}{3} M_m - (tg\, \alpha + tg\, \beta) \frac{E J_1}{2 l_1}, \right]$ where $M_m$ is the

max. moment of girder $A, B$, considered as a free beam. We get from above also:

10. $\frac{M_1}{M_2} = \frac{P_1 k_1 + P_2 k_2 + V_0 k_0}{Q_1 r_1 + Q_2 r_2 + V_3 k_3}.$

In the 9 equations, 1 to 9 are contained the 10 unknown values: $P_1, P_2, V_0, Q_1, Q_2, V_3, M_1, M_2, \alpha$ and $\beta$.

As the tenth equation is a resultant from the nine first, we cannot find the true values of the unknown quantities without further assumptions for our general case. These, however, will be reserved to the designer, and we can find from the above some relations, which will help to proportion the sections of the different members.

Introducing equations 1, 2, 3 in 4, also 5, 6, 7 in 8, and the resultant equations in 9, we get:

$$M_1 + M_2 = \frac{4}{3} M_m - 2 \,(tg\, \alpha + tg\, \beta) \frac{E J_1}{2 l_1}$$
$$= 2 E \, tg\, \alpha \,(J_\alpha' + J_\alpha'' + J_0) + 2 E \, tg\, \beta \,(J\beta' + J\beta'' + J_3)$$

If we call, for shortness, the constant values:

$$J_\alpha' + J_\alpha'' + J_0 = Z_1$$
$$J\beta' + J\beta'' + J_3 = Z_2,$$ we get:

$$\tfrac{4}{8} M_m = tg\,\alpha \times \frac{E J_1}{l_1} + tg\,\beta \frac{E J_1}{l_1} + 2\,E\,tg\,\alpha\,Z_1 + 2\,E\,tg\,\beta\,Z_2$$

$$tg\,\beta \left(\frac{E J_1}{l_1} + 2\,E\,Z_2\right) = \tfrac{4}{8} M_m - tg\,\alpha \left(\frac{E J_1}{l_1} + 2\,E\,Z_1\right)$$

$$tg\,\beta = \frac{\tfrac{4}{8} M_m - tg\,\alpha \left(\frac{E J_1}{l_1} + 2\,E\,Z_1\right)}{\frac{E J_1}{l_1} + 2\,E\,Z_2}$$

From equation 10 we get:

$$\frac{M_1}{M_2} = \frac{Z_1\,tg\,\alpha}{Z_2\,tg\,\beta}, \text{ or:}$$

$$tg\,\beta = tg\,\alpha \frac{M_2 Z_1}{M_1 Z_2}; \text{ therefore:}$$

$$tg\,\alpha \frac{M_2 Z_1}{M_1 Z_2} \left[\left(\frac{E J_1}{l_1} + 2\,E\,Z_2\right) + \left(\frac{E J_1}{l_1} + 2\,E\,Z_1\right)\right] = \tfrac{4}{8} M_m$$

$$\text{11.} \quad tg\,\alpha = \frac{\tfrac{4}{8} M_m}{\frac{M_2 Z_1}{M_1 Z_2}\left(\frac{E J_1}{l_1} + 2\,E\,Z_2\right) + \frac{E J_1}{l_1} + 2\,E\,Z_1}$$

Again, according to equation 4:

$M_1 = P_1 k_1 + P_2 k_2 + V_0 k_0$, or, with 1, 2, 3:

$M_1 = 2\,E\,tg\,\alpha\,(J_a' + J_a'' + J_0) = 2\,E\,tg\,\alpha\,Z_1$; with 11:

$$M_1 = \frac{\tfrac{4}{3} M_m \times 2\,E\,Z_1}{\frac{M_2 Z_1}{M_1 Z_2}\left(\frac{E J_1}{l_1} + 2\,E\,Z_2\right) + \frac{E J_1}{l_1} + 2\,E\,Z_2}, \text{ or,}$$

$$\text{12.} \quad M_1 = \frac{8}{3} \times \frac{M_m Z_1}{\frac{M_2 Z_1}{M_1 Z_2}\left(\frac{J_1}{l_1} + 2\,Z_2\right) + \frac{J_1}{l_1} + 2\,Z_1}$$

Likewise we find:

$$\text{13.} \quad tg\,\beta = \frac{\tfrac{4}{3} M_m}{\frac{M_1 Z_2}{M_2 Z_1}\left(\frac{E J_1}{l_1} + 2\,E\,Z_1\right) + \frac{E J_1}{l_1} + 2\,E\,Z_2}$$

$$\text{14.} \quad M_2 = \frac{8}{3} \times \frac{M_m Z_2}{\frac{M_1 Z_2}{M_2 Z_1}\left(\frac{J_1}{l_1} + 2\,Z_1\right) + \frac{J_1}{l_1} + 2\,Z_2}$$

Introducing the values of $tg\,\alpha$ and $tg\,\beta$—equations 11 and 13—in 1, 2, 3, 5, 6, 7, we can easily determine all unknown quantities, if $tg\,\alpha$, $tg\,\beta$, $M_1$ and $M_2$, are known. Equations 11 to 14 contain the still undetermined factor $\frac{M_1}{M_2}$, or the relation between the real moments at the ends of the girder. But $M_1$ and $M_2$ are in reversed proportion with the respective unit strains to be used in $A$ and $B$ for the columns and girders at these points, and in direct proportion with the moments of resistance, (or the moments of inertia divided by the half average depth $c_1$ and $c_2$ of the columns and girders in points $A$ and $B$.) It is reasonable for ordinary cases to assume the max. unit strains in $A$ and $B$ from live load to be alike. Then we have the simple proportion :

$$\frac{M_1}{M_2} = \frac{Z_1 \times c_2}{Z_2 \times c_1},$$

which is to be substituted in the above formulae for practical use. This will give only approximate but direct results. It will be seen that the designer is free to provide for another relation between $M_1$ and $M_2$, and can easily check the consequences by means of above formulaes, which allow more than one resolution. If the judgment in a special case allows the assumption that,

$$M_1 = M_2 \text{ and } Z_2 = Z_1,$$

we get from equation 12 the simpler expression :

$$M_1 = M_2 = M = \frac{4}{3} \times \frac{M_m Z}{\frac{J_1}{l_1} + 2Z};$$

which gives direct and exact results.

By means of these formulaes it will be very easy to proportion the different members of a bent, such as shown, for instance, in Fig. 8, taking in account the dead load, (which will be a much simpler problem). Having determined the sizes of the members to suit these formulaes, it will not be a very troublesome task to ascertain the results by constructing the elastic lines of the columns and girders. These have to fulfill the conditions proposed by the designer.